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ON HOOKE'S LAW*

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A novel method of describing elastic anisotropy based on the concept of an elastic eigen state is proposed. The structure of the rigidity tensor is determined. In particular, it is shown that the set of 21 constants describing continuously the manifold of elastic solids is composed of three different subsystems: 6 true rigidity moduli, 12 dimensionless rigidity distributors, and 3 invariant parameters defining the orientation of the body in question relative to the laboratory system of coordinates. It is shown that Hooke's Law can be written uniquely for an arbitrary anisotropic body in the form of several laws describing the direct proportionality of the corresponding parts of the stress and deformation tensors. The construction of these parts is illustrated using an example of a transversally isotropic solid.

1. We shall assume that the deformations are measured from the natural, stress-free state and are small, the influence of the temperature and other fields is insignificant, and the stress tensor σ is symmetric and is a linear reversible function of the deformation tensor ε

$$\sigma = \mathbf{C}\cdot\varepsilon, \ \varepsilon = S\cdot\sigma \tag{1.1}$$

The elastic behaviour is fully described by the material rigidity tensor $\mathbf{C}$, or by the compliance tensor S. One is the inverse of the other, i.e.

$$\mathbf{C}\circ S = S\circ \mathbf{C} = \mathbf{I}, \quad \mathbf{I}\cdot\alpha \equiv \alpha \tag{1.2}$$

The law (1.1) was discovered in its basic form by Hooke /1/. The form of the tensor C was obtained for an isotropic solid during the first quarter of the last century /2/

$$\mathbf{C} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I} \tag{1.3}$$

The form of C for all crystal classes was more or less obtained at the beginning of this country /3, 4/. The monographs /5—12/ can be used as a source of results of the determination of the elastic constants of specific materials and the quantitative features of the behaviour of anisotropic elastic solids and of the calculations of the effective constants for complex bodies.

At the same time we find, that many new materials, especially composites, have very anisotropic elastic properties, up to complete lack of symmetry.

In the latter case, which also corresponds to that of triclinic crystals, group theory has "nothing to latch onto" and the existing approach yields no information whatsoever on the rigidity tensor. The theory leaves the experimenter no choice, but to measure 21 components C_{ijkl} on a random basis.

The cardinal problem of constructing efficiently a complete system of 18 invariants of the rigidity tensor, remains unsolved. In other words, we have no general algorithm for identifying two rigidity tensors in terms of the sets of their components measured in two different laboratories. The elegant proposition that the intrinsic bases should be chosen in terms of the convolutions of C_{ijkk} and C_{ikjk} /13/, is a substantial advance towards solving the problem, but is unsuitable in the case when the tensors have a symmetry axis, or are, altogether isotropic.

Important though the symmetry might be, the laws need not be developed only in terms of it. We shall show that Hooke's Law conceals within it quite different possibilities for describing the structure of an arbitrary, linearly elastic solid.

2. We shall call the tensor $\mathbf{C}$ the elastic solid $\mathbf{C}$, with significance (which can easily be made more precise). The essence of the proposed method of describing elastic properties lies in the following. Let us take any elastic solid $\mathbf{C}$ and consider its exclusive states in which the stress and deformation tensors are not only coaxial, but also strictly proportional

$$\sigma = \lambda\varepsilon \tag{2.1}$$

Definition. The parameter λ which has the dimensions of stress, will be called the true rigidity of the elastic solid $\mathbf{C}$, provided that a symmetric second rank tensor ω exists, satisfying the equation

$$\mathbf{C}\cdot\omega = \lambda\omega \tag{2.2}$$

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We shall call the tensor $\boldsymbol{\omega}$ the elastic eigen state of the solid $\mathbf{C}$, corresponding to the true rigidity modulus λ.

From (1.2) it follows that when $\lambda \neq 0$, the equation (2.2) is equivalent to

$$\mathbf{S}\cdot\boldsymbol{\omega} = \frac{1}{\lambda}\boldsymbol{\omega} \tag{2.3}$$

The elastic eigen state ω can be regarded either as a special deformed state, or as a special stress state.

It is natural to expect from the mathematical, as well as the physical point of view, that the set of all true rigidity moduli and the corresponding elastic eigen states describes completely and in depth the behaviour of an elastic solid. We shall prove this, demanding only that several facts of the theory of linear operators be expressed in terms of tensor notation.

3. Let us denote by $\mathcal{G}$ the initial three-dimensional vector space, and by $O = O(3)$ the group of its orthogonal transformations. The space of symmetric tensors of second rank over $\mathcal{G}$ will be denoted by Σ, and its tensor square by T

$$\Sigma \equiv \operatorname{sym} \mathcal{G} \underset{\sim}{\otimes} \mathcal{G}, \quad T \equiv \Sigma \otimes \Sigma \tag{3.1}$$

Thus we have by definition $\alpha_{ij} = \alpha_{ji}$ for any $\alpha \in \Sigma$ and $L_{ijkl} = L_{jikl} = L_{ijlk}$ for any $\mathbf{L} \in T$. Clearly, $\mathbf{C} \in T$.

Let us introduce into Σ the usual (energy) scalar product

$$(\alpha, \beta) \to \alpha\cdot\beta \equiv \alpha_{ij}\beta_{ij} \tag{3.2}$$

The product is compatible with the Euclidean tensor structure in Σ, i.e. it is invariant with respect to the group O. The set Σ together with the operations of tensor summation, multiplication of a tensor by a number and of scalar multiplication (3.2), represents a 6-dimensional Euclidean space.

Henceforth, we shall utilise the orthonormed bases in this space, i.e. the sextuplets of tensors

$$\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots, \boldsymbol{\omega}_6, \quad \boldsymbol{\omega}_K\cdot\boldsymbol{\omega}_L = \delta_{KL} \equiv \begin{cases} 1 & K = L \\ 0 & K \neq L \end{cases} \tag{3.3}$$

Here and henceforth the capital Latin indices $K, L, \ldots$ run through the values $1, 2, \ldots, 6$, and the Elinstein summation rule does not apply to them. We shall write the expansion of any tensor α in terms of the basis elements (3.3) in the form

$$\boldsymbol{\alpha} = \alpha_1\boldsymbol{\omega}_1 + \ldots + \alpha_6\boldsymbol{\omega}_6, \quad \alpha_K = \boldsymbol{\alpha}\cdot\boldsymbol{\omega}_K \tag{3.4}$$

Any tensor $\mathbf{L} \in T$ can be identified with the linear operator

$$\boldsymbol{\alpha} \to \mathbf{L}\cdot\boldsymbol{\alpha} \tag{3.5}$$

transforming the space Σ into itself. From (3.4) we have

$$\mathbf{L}\cdot\boldsymbol{\alpha} = \alpha_1\mathbf{L}\cdot\boldsymbol{\omega}_1 + \ldots + \alpha_6\mathbf{L}\cdot\boldsymbol{\omega}_6 = (\mathbf{L}\cdot\boldsymbol{\omega}_1\otimes\boldsymbol{\omega}_1 + \ldots + \mathbf{L}\cdot\boldsymbol{\omega}_6\otimes\boldsymbol{\omega}_6)\cdot\boldsymbol{\alpha} \tag{3.6}$$

for any $\alpha \in \Sigma$.

This leads (only because both fourth rank tensors in (3.6) belong to the tensor subspace T) to a fundamental tensor identity: for any tensor $\mathbf{L} \in T$ and any orthonormed basis $\boldsymbol{\omega}_K$

$$\mathbf{L} = \mathbf{L}\cdot\boldsymbol{\omega}_1\otimes\boldsymbol{\omega}_1 + \ldots + \mathbf{L}\cdot\boldsymbol{\omega}_6\otimes\boldsymbol{\omega}_6 \tag{3.7}$$

The identity expresses explicitly the linear operator (3.5) in terms of its values on the basis (3.3), and is equivalent to the following identity:

$$\mathbf{I} = \boldsymbol{\omega}_1\otimes\boldsymbol{\omega}_1 + \ldots + \boldsymbol{\omega}_6\otimes\boldsymbol{\omega}_6 \tag{3.8}$$

Indeed, (3.8) follows from (3.7), since $\mathbf{I}\cdot\boldsymbol{\alpha} = \boldsymbol{\alpha}$ for any $\boldsymbol{\alpha} \in \Sigma$. Conversely, (3.7) follows from (3.8) since $\mathbf{L}\circ\mathbf{I} = \mathbf{I}\circ\mathbf{L} = \mathbf{L}$ for any $\mathbf{L} \in T$.

According to the definition of the tensor product of linear spaces, the set of 36 tensors $\boldsymbol{\omega}_K\otimes\boldsymbol{\omega}_L$ forms a basis in $T \equiv \Sigma\otimes\Sigma$. Therefore any tensor $\mathbf{L} \in \mathrm{T}$ can be written uniquely in the form

$$\mathbf{L} = \sum_{K,L=1}^{6} L_{KL}\boldsymbol{\omega}_K\otimes\boldsymbol{\omega}_L, \quad L_{KL} \equiv \boldsymbol{\omega}_K\cdot\mathbf{L}\cdot\boldsymbol{\omega}_L \tag{3.9}$$

Henceforth, the orothogonal projectors play a decisive role. Let us consider the subspace $\Pi \subset \Sigma$ and its orthogonal complement $\Pi^\perp$ The formula

$$\Sigma = \Pi \oplus \Pi^\perp \tag{3.10}$$

means, as usual, that for any tensor $\alpha \in \Sigma$ there are precisely two tensors α_Π and $\alpha_\Pi{}^\perp$ such that

$$\alpha = \alpha_\Pi + \alpha_\Pi^\perp, \quad \alpha_\Pi \cdot \alpha_\Pi^\perp = 0, \quad \alpha_\Pi \in \Pi \tag{3.11}$$

The tensor α_Π is an orthogonal projection of α on Π. The tensor $\mathbf{P} \in T$, uniquely defined by the condition

$$\mathbf{P} \cdot \alpha = \alpha_\Pi, \quad \forall \alpha \in \Sigma \tag{3.12}$$

will be called the orothogonal projector on the subspace Π. If a part $M \equiv (\omega_{U+1}, \ldots, \omega_{U+V})$ of the basis $\omega_1, \ldots, \omega_6$ is situated in Π, then $\mathbf{P} \cdot \omega_S = \omega_S$ for $\omega_S \in M$ and $\mathbf{P} \cdot \omega_S = 0$ for $\omega_S \notin M$, and according to (3.9) we obtain

$$\mathbf{P} = \omega_{U+1} \otimes \omega_{U+1} + \ldots + \omega_{U+V} \otimes \omega_{U+V} \tag{3.13}$$

From this we have $V = \dim \Pi = P_{ijij}$.

Two orthogonal projectors, $\mathbf{P}_1$ on Π_1 and $\mathbf{P}_2$ on Π_2, will be called orthogonal if the subspaces Π_1, Π_2 are mutually orthogonal. This is equivalent to the tensor equation

$$\mathbf{P}_1 \circ \mathbf{P}_2 = \mathbf{P}_2 \circ \mathbf{P}_1 = \mathbf{0}$$

The system of pairwise mutually orthogonal projectors $\mathbf{P}_1, \ldots, \mathbf{P}_\rho$ will be called an orthogonal expansion of unity, provided that

$$\mathbf{I} = \mathbf{P}_1 + \ldots + \mathbf{P}_\rho \tag{3.14}$$

Any expansion of unity can be obtained by choosing the corresponding basis ω_K and grouping the terms in (3.8) in the corresponding manner. The expansion of unity (3.14) is in one to one correspondence with the expansion of the space Σ into the right sum of the subspaces $\Pi_\alpha \equiv \mathrm{Im}\, \mathbf{P}_\alpha$, $\alpha = 1, \ldots, \rho$.

4. We shall consider only hyperelastic bodies. In other words, we will assume that the body $\mathbf{C}$ has an elastic potential

$$2\Phi(\varepsilon) = \sigma \cdot \varepsilon = \varepsilon \cdot \mathbf{C} \cdot \varepsilon, \quad \sigma = \partial_\varepsilon \Phi \tag{4.1}$$

This is equivalent to the condition of symmetry with respect to the scalar product (3.2)

$$\alpha \cdot \mathbf{C} \cdot \beta = \beta \cdot \mathbf{C} \cdot \alpha, \quad \forall \alpha, \beta \in \Sigma \tag{4.2}$$

Thus we deal with the linear symmetric operator $\alpha \to \mathbf{C} \cdot \alpha$ acting in a finite-dimensional space with a scalar product. (Strictly speaking, the stresses should be referred to a fixed standard; the reader can assume that this has been done). The situation has been investigated as fully as possible, and it only remains to translate the information available into the language of mechanics.

The spectral theorem takes the form of the following fundamental structural theorem.

Theorem 1. For any elastic solid $\mathbf{C}$ there exists exactly one orthogonal expansion of the space of symmetric tensors of second rank

$$\Sigma = \Pi_1 \oplus \ldots \oplus \Pi_\rho, \quad \Pi_\alpha \perp \Pi_\beta \text{ for } \alpha \neq \beta, \ \rho \leqslant 6 \tag{4.3}$$

and exactly one set of true moduli of rigidity

$$\lambda_1, \ldots, \lambda_\rho, \quad \lambda_\alpha \neq \lambda_\beta \text{ for } \alpha \neq \beta \tag{4.4}$$

such that

$$\mathbf{C} = \lambda_1 \mathbf{P}_1 + \ldots + \lambda_\rho \mathbf{P}_\rho \tag{4.5}$$

where $\mathbf{P}_\alpha$ is an orthogonal projector on Π_α, $\alpha = 1, \ldots, \rho$.

The proof can be obtained from /14–16/ etc. by suitable changing the terminology.

The mechanical meaning of the terms Π_α is clear: the subspace Π_α consists of all elastic eigenstates corresponding to the modulus of rigidity λ_α. Indeed, we have for any $\omega \in \Pi_\alpha$

$$\mathbf{C} \cdot \omega = (\lambda_1 \mathbf{P}_1 + \ldots + \lambda_\rho \mathbf{P}_\rho) \cdot \omega = \lambda_\alpha \omega \tag{4.6}$$

Thus we have a formula giving in explicit form the rigidity tensor in terms of its true moduli of rigidity and elastic eigenstates.

Let us give other equivalent formulations of the fundamental structural theorem, which may perhaps prove more suitable.

1°. For any elastic solid $\mathbf{C}$ at least one orthonormed basis ω_K exists in Σ consisting of elastic eigenstates of this solid

$$\mathbf{C} \cdot \omega_K = \lambda_K \omega_K, \quad K = 1, \ldots, 6 \tag{4.7}$$

2°. For any elastic solid $\mathbf{C}$ at least one orthonormed basis ω_K exists in Σ and six parameters $\lambda_1, \ldots, \lambda_6$ such that *

$$\mathbf{C} = \lambda_1 \boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_1 + \ldots + \lambda_6 \boldsymbol{\omega}_6 \otimes \boldsymbol{\omega}_6 \tag{4.8}$$

3° For any elastic solid $\mathbf{C}$ at least one orthonormed basis $\boldsymbol{\omega}_K$ exists in Σ such that the matrix

$$C_{KL} \equiv \boldsymbol{\omega}_K \cdot \mathbf{C} \cdot \boldsymbol{\omega}_L \tag{4.9}$$

is diagonal.

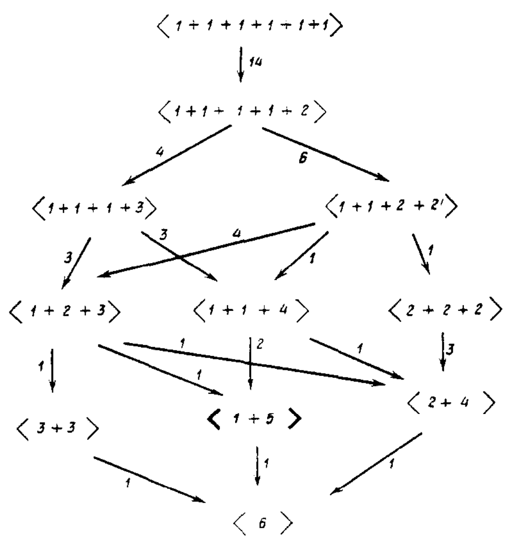

Fig.1

It will be helpful to observe the equivalence of the representations (4.5), (4.8). Formula (4.8) follows from (4.5) by virtue of (3.13). Conversely, taking into account in (4.8) all congruences λ_K and using (3.13), we arrive at (4.5), all terms of the sum being unique. For $\rho = 6$ the formulas (4.5), (4.8) are simply identical, with the one-dimensional projectors

$$\mathbf{P}_1 = \boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_1, \ldots, \mathbf{P}_6 = \boldsymbol{\omega}_6 \otimes \boldsymbol{\omega}_6 \tag{4.10}$$

We shall call (4.5) or (4.8) the fundamental structural formula for an elastic solid. Any orthonormed set of the eigenstates $\boldsymbol{\omega}_K$ of the tensor $\mathbf{C}$ will be called the material tensor reference point of the body in question.

For the expansion (4.3) a corresponding decomposition exists of the number six into the integral positive terms

$$q_\alpha \equiv \dim \Pi_\alpha = P_{(\alpha)ijij} \tag{4.11}$$

(or, if preferred, Young's scheme /18/). We shall write this decomposition in the form

$$\langle q_1 + \ldots + q_\rho \rangle, \quad q_1 \leqslant \ldots \leqslant q_\rho \tag{4.12}$$

We shall call the set of all elastic bodies with the same first structural index the elastic form. We shall say that the elastic form $\langle k_1 + \ldots + k_t \rangle$ is subordinate to the elastic form $\langle m_1 + \ldots + m_u \rangle$ if $t < u$ and k_i are either equal to some m_i, or are their sums. Fig.1 shows the scheme of subordination of all 11 possible elastic forms. One form is subordinate to another, provided that one can pass from the latter to the former along the arrows shown. Each level of the scheme consists of all elastic solids with the same number of pairwise different true moduli of rigidity. The passage from the k-th level to the $(k-1)$-th level is made by making two moduli identical. The numbers accompanying the arrows indicate the number of possible identifications.

We stress the fact that elastic solids of one elastic form can differ considerably from each other in the nature of the elastic eigenstates and in this symmetry properties, since the first structural index takes into account only the dimensionality of the spaces of eigenstates.

5. The rigidity tensor given in the form (4.5) or (4.8) can be inverted quite rapidly. According to (1.2) the eigenelements of $\mathbf{C}$ and $\mathbf{S}$ are the same and the eigenvalues are invertible. Consequently the structural formula for the compliance tensor has the form

$$\mathbf{S} = \frac{1}{\lambda_1}\mathbf{P}_1 + \ldots + \frac{1}{\lambda_\rho}\mathbf{P}_\rho = \frac{1}{\lambda_1}\boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_1 + \ldots + \frac{1}{\lambda_6}\boldsymbol{\omega}_6 \otimes \boldsymbol{\omega}_6 \tag{5.1}$$

The quantities λ_α^{-1} represent the true compliance moduli.

6. If the rigidity tensor $\mathbf{C}$ is given in terms of its components C_{ijkl} on some basis, then the rigidity moduli λ_α and elastic eigenstates can be found as follows. We take in Σ any orthonormalized basis $\boldsymbol{\tau}_P$, $\boldsymbol{\tau}_P \cdot \boldsymbol{\tau}_Q = \delta_{PQ}$ and introduce the matrix

$$C_{PQ} \equiv \boldsymbol{\tau}_P \cdot \mathbf{C} \cdot \boldsymbol{\tau}_Q = \tau_{(P)ij} C_{ijkl} \tau_{(Q)kl} \tag{6.1}$$

By (2.2) the rigidity moduli will be roots of the sixth-degree equation

$$\det\,(C_{PQ} - \lambda\delta_{PQ}) = 0 \tag{6.2}$$

*) The concept given here was presented in the course of lectures given by the author since the end of the sixties in a number of Polish and Soviet research centres. A special case of (4.8) was used effectively in /17/. (See also: Rychlewski J. Mathematical structure of elastic solids. Preprint of Inst. of Problems of Mechanics. AS SSSR, No.217, Moscow, 1983).

It can be shown that the coefficient of this equation are independent of the choice of the basis τ_K, i.e. are invariants of the tensor $\mathbf{C}$*

The multiplicity of the root λ_α is equal to the dimensions q_α of the space Π_α

The projectors of the structural formula (4.6) are also obtained using standard methods /14/

$$\mathbf{P}_\alpha = \frac{1}{\Delta}[(\mathbf{C}-\lambda_1\mathbf{I})\circ\ldots\circ(\mathbf{C}-\lambda_{\alpha-1}\mathbf{I})\circ(\mathbf{C}-\lambda_{\alpha+1}\mathbf{I})\circ\ldots\circ(\mathbf{C}-\lambda_\rho\mathbf{I})] \tag{6.3}$$

$$\Delta \equiv (\lambda_\alpha-\lambda_1)\ldots(\lambda_\alpha-\lambda_{\alpha-1})(\lambda_\alpha-\lambda_{\alpha+1})\ldots(\lambda_\alpha-\lambda_\rho) \quad 1\leqslant\alpha\leqslant\rho \tag{6.4}$$

We note the formulas

$$\lambda_\alpha = \mathbf{C}\cdot\mathbf{P}_\alpha = C_{ijkl}P_{(\alpha)ijkl} \tag{6.5}$$

Incidently, it is obvious that (6.2) and (6.3) will not be used all that often. In principle, the elastic solid can be described directly by the set of rigidity moduli and elastic eigenstates, i.e. by (4.5) or (4.8).

7. Let us decompose the stress and deformation tensors over the spaces of elastic eigenstates of the solid in question

$$\begin{gathered}\boldsymbol{\sigma}=\boldsymbol{\sigma}_1+\ldots+\boldsymbol{\sigma}_\rho,\quad \boldsymbol{\sigma}_\alpha\equiv\mathbf{P}_\alpha\cdot\boldsymbol{\sigma}\in\Pi_\alpha\\ \boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_1+\ldots+\boldsymbol{\varepsilon}_\rho,\quad \boldsymbol{\varepsilon}_\rho\equiv\mathbf{P}_\alpha\cdot\boldsymbol{\varepsilon}\in\Pi_\alpha\end{gathered} \tag{7.1}$$

$$\boldsymbol{\sigma}_\alpha\cdot\boldsymbol{\sigma}_\beta=0,\quad \boldsymbol{\varepsilon}_\alpha\cdot\boldsymbol{\varepsilon}_\beta=0,\quad \boldsymbol{\sigma}_\alpha\cdot\boldsymbol{\varepsilon}_\beta=0 \text{ for } \alpha\neq\beta \tag{7.2}$$

The basic structural theorem can be expressed in the following form: Hooke's Law can be decomposed in a unique manner into a system of $\rho\leqslant 6$ independent, mutually orthogonal laws of direct proportionality

$$\boldsymbol{\sigma}_1=\lambda_1\boldsymbol{\varepsilon}_1,\ldots,\boldsymbol{\sigma}_\rho=\lambda_\rho\boldsymbol{\varepsilon}_\rho \tag{7.3}$$

for any elastic solid $\mathbf{C}$

Indeed, substituting (7.1) and the structural formula (4.5) into (1.1) we obtain (7.3). Conversely, summing (7.3) we obtain (4.5).

Equation (7.3) with index α is understood to be equivalent to q_α scalar equations. For $\rho=6$ all $q_\alpha=1$ (the roots of (6.2) are simple) and Hooke's Law can be written in the form of six scalar equations

$$\begin{gathered}\sigma_1=\lambda_1\varepsilon_1,\ \ldots,\ \sigma_6=\lambda_6\varepsilon_6\\ \sigma_K\equiv\boldsymbol{\sigma}\cdot\boldsymbol{\omega}_K,\ \varepsilon_K=\boldsymbol{\varepsilon}\cdot\boldsymbol{\omega}_K\end{gathered} \tag{7.4}$$

Hooke's Law in the form (7.3), and even more so (7.4), seems to have reverted to the initial formulation given by the author himself in his remarkable anagram ceiiinosssttuv (ut tensio sic vis) /1/.

Let us introduce the intensities

$$s_\alpha\equiv|\boldsymbol{\sigma}_\alpha|=(\boldsymbol{\sigma}\cdot\mathbf{P}_\alpha\cdot\boldsymbol{\sigma})^{1/2},\quad e_\alpha\equiv|\boldsymbol{\varepsilon}_\alpha| \tag{7.5}$$

so that

$$|\boldsymbol{\sigma}|^2=s_1^2+\ldots+s_\rho^2,\quad |\boldsymbol{\varepsilon}|^2=e_1^2+\ldots+e_\rho^2 \tag{7.6}$$

From (7.3) there follows the proportionality of the intensities

$$s_1=\lambda_1e_1,\ \ldots,\ s_\rho=\lambda_\rho e_\rho \tag{7.7}$$

The decomposition (7.3) generalises the representation of Hooke's Law given in textbooks for an isotropic body in the form of two tensor equations: the law of proportionality of the spherical and the deviator parts of the tensors $\boldsymbol{\sigma},\boldsymbol{\varepsilon}$ (See Sect.15 below).

8. Let us now consider the elastic energy. Substituting (7.1) into (4.1), we obtain

$$2\Phi=s_1e_1+\ldots+s_\rho e_\rho$$

We see that if s_α is treated as a generalised thermodynamic force, then e_α will represent the corresponding generalised thermodynamic coordinate. Substituting (7.3) here we obtain

$$2\Phi=\lambda_1e_1^2+\ldots+\lambda_\rho e_\rho^2=\frac{1}{\lambda_1}s_1^2+\ldots+\frac{1}{\lambda_\rho}s_\rho^2$$

from which we obtain

Theorem 2. The elastic energy of a solid $\mathbf{C}$ is positive for any state of deformation $\boldsymbol{\varepsilon}\neq 0$ if and only if its true rigidity moduli are positive $\lambda_1>0,\ldots,\lambda_\rho>0$.

Note the simplicity of these conditions.

*) The eigenvalues of the matrix C_{PQ} were investigated, for some set τ_K, by K.S. Aleksandrov in the review: Elastic properties of anisotropic media, Doctorate Dissertation, Inst. Crystallography, Academy of Sciences of the USSR, Moscow, 1967.

Using any material reference tensor ω_K, we obtain

$$2\Phi = \lambda_I e_I^2 + \ldots + \lambda_6 e_6^2 = \frac{1}{\lambda_I}\sigma_I^2 + \ldots + \frac{1}{\lambda_6}\sigma_6^2$$

The above representation of the elastic energy lends itself to the following geometrical treatment: the isoenergetic surface $2\Phi(\sigma) = 1$ is a 6-dimensional ellipsoid in Σ and the axes of this ellipsoid are directed along the elastic eigenstates ω_K, while the lengths of the semiaxes are equal to the roots of true rigidity moduli $\sqrt{\lambda_K}$.

9. We shall consider the problem of determining the system of independent scalar parameters describing continuously the manifold of elastic solids.

According to the basic structural formula (4.8), any set

$$(\lambda_I, \ldots, \lambda_6; \omega_I, \ldots, \omega_6) \tag{9.1}$$

consisting of positive λ_K and an orthonormalized basis ω_K, defines some, theoretically possible elastic solid, for which λ_K will represent the true rigidity moduli and ω_K the material tensor basis. (Experiments show that the elastic constants satisfy additional natural phenomenological restrictions which do not follow from thermodynamic concepts, such as constraints imposed on Poisson's ratio, etc.; we shall put this interesting problem to one side).

Our problem reduces to that of determining a set of free parameters describing continuously the manifold of orthonormalised bases in Σ.

As an example of the system of 36 parameters describing continuously, in general, six symmetric tensors ω_K relative to the laboratory coordinate system, we can take e.g. the following set:

$$\operatorname{tr}\omega_K,\ \operatorname{tr}\omega_K^2,\ \operatorname{tr}\omega_K^3,\ \ \theta_K,\ \varphi_K,\ \psi_K \tag{9.2}$$

Here θ_K, φ_K, ψ_K denote the Eulerian angles of the principal axes of the tensor ω_K relative to the laboratory coordinate system.

Six normalizing conditions $\omega_K \cdot \omega_K \equiv \operatorname{tr}\omega_K^2 = 1$ eliminate $\operatorname{tr}\omega_K^2$, and 15 conditions of orthogonality are imposed on the remaining 30 parameters, $\omega_K \cdot \omega_L = 0$ for $K \neq L$. Three angles $\theta_L, \varphi_L, \psi_L$ can be included in the system for some L. The remaining 12 independent parameters $\varkappa_1, \ldots, \varkappa_{12}$ will form a system of independent invariants of the set $\omega_I, \ldots, \omega_6$.

We have the following result:

the manifold of elastic solids with elastic potential is described continuously, in the general case, by a system of 21 parameters consisting of the following subsystems, differing considerably for each other:

6 invariants of the rigidity tensor

$$\lambda_I, \ldots, \lambda_6 \tag{9.3}$$

with dimensions of stress, and positive. The invariants determine the degree of general rigidity of the body, and are called true rigidity moduli. The corresponding true compliance moduli are

$$\lambda_I^{-1}, \ldots, \lambda_6^{-1} \tag{9.4}$$

12 dimensionless invariants of the rigidity tensor

$$\varkappa_1, \ldots, \varkappa_{12} \tag{9.5}$$

forming a functionally complete and irreducible system of invariants of the material tensor reference frame ω_K. They are identical forthe rigidity and the compliance tensor. We shall call these constants the rigidity distributors.

3 invariant parameters, e.g. the Euler angles θ, φ, ψ fixing the elastic solid in question relative to the laboratory coordinate system.

A system of 18 invariants of the rigidity tensor

$$(\lambda_I, \ldots, \lambda_6, \varkappa_1, \ldots, \varkappa_{12})$$

can be called the system of elastic material constants. (We note that it is generally incorrect to call the components C_{ijkl} elastic constants in a random basis.

In choosing the system of independent invariants $\varkappa_1, \ldots, \varkappa_{12}$ we should remember that the invariants (9.2) are connected by relations such as e.g. the relation

$$(\operatorname{tr}\omega_I)^2 + \ldots + (\operatorname{tr}\omega_6)^2 = 3 \tag{9.6}$$

which follows at once from (3.8).

This is the situation in the "general case", i.e. in the neighbourhood of $\mathbf{C} \in T$ chosen in a suitable manner. For a particular family of elastic solids the number of pairwise different rigidity moduli is equal to ρ, and we have k rigidity distributors and f orientation parameters. We shall call the symbol

$$[\rho + k + f], \quad \rho \leqslant 6,\ k \leqslant 12,\ f \leqslant 3 \tag{9.7}$$

the second structural index of this family.

10. Let us use the set (9.1) to describe certain specific elastic properties. From (4.8) we obtain at once the following elegant formulas:

$$\mathrm{Tr}\,\mathbf{C} \equiv C_{ijij} = \lambda_I + \ldots + \lambda_6 = q_1\lambda_1 + \ldots + q_\rho\lambda_\rho \tag{10.1}$$

$$\mathbf{C}\cdot\mathbf{C} \equiv C_{ijkl}C_{ijkl} = \lambda_I^2 + \ldots + \lambda_6^2 = q_1\lambda_1^2 + \ldots + q_\rho\lambda_\rho^2 \tag{10.2}$$

Thus the invariant $\mathrm{Tr}\,\mathbf{C}/6$ has the sense of the mean rigidity modulus, and the invariant $(\mathbf{C}\cdot\mathbf{C})^{1/2}/\sqrt{6}$ of the root mean square rigidity modulus.

The modulus of uniform compression K is given by the expression

$$\frac{1}{K} = \frac{(\mathrm{tr}\,\omega_I)^2}{\lambda_I} + \ldots + \frac{(\mathrm{tr}\,\omega_6)^2}{\lambda_6} \tag{10.3}$$

Young's modulus in the direction $\mathbf{n}$, $E(\mathbf{n})$ is given by

$$\frac{1}{E(\mathbf{n})} = \frac{(\mathbf{n}\omega_I\mathbf{n})^2}{\lambda_I} + \ldots + \frac{(\mathbf{n}\omega_6\mathbf{n})^2}{\lambda_6} \tag{10.4}$$

Poisson's ratio $\nu(\mathbf{n}, \mathbf{m})$ in the $\mathbf{m}$ direction under tension and in the $\mathbf{n}$ direction under compression, is equal to

$$-\frac{\nu(\mathbf{n}, \mathbf{m})}{E(\mathbf{n})} = \frac{(\mathbf{n}\omega_I\mathbf{n})(\mathbf{m}\omega_I\mathbf{m})}{\lambda_I} + \ldots + \frac{(\mathbf{n}\omega_6\mathbf{n})(\mathbf{m}\omega_6\mathbf{m})}{\lambda_6} \tag{10.5}$$

The shear modulus $G(\mathbf{n}, \mathbf{m})$, $\mathbf{nm} = 0$ in the case of a simple shear in the plane containing $\mathbf{n}$, $\mathbf{m}$, is given by the expression

$$\frac{1}{4G(\mathbf{n}, \mathbf{m})} = \frac{(\mathbf{n}\omega_I\mathbf{m})^2}{\lambda_I} + \ldots + \frac{(\mathbf{n}\omega_6\mathbf{m})^2}{\lambda_6} \tag{10.6}$$

11. If a solid, e.g. a composite, is constructed in such a manner that one of the moduli λ_ν is much greater than the others, we can idealize the rigid constraint

$$\lambda_\nu^{-1} = 0, \quad 1 \leqslant \nu \leqslant \rho \tag{11.1}$$

In this case $\varepsilon_\nu \equiv \mathbf{P}_\nu\cdot\varepsilon = 0$ and the right-hand side of the ν-th equation of Hooke's Law (7.3) will represent an indeterminacy of the type $\infty\cdot 0$. For this reason the part σ_ν of the stress tensor can be regarded as a reaction and is therefore not defined by Hooke's Law. The rigidity and compliance tensors should be taken in the form

$$\mathbf{C} = \lambda_1\mathbf{P}_1 + \ldots + \lambda_{\nu-1}\mathbf{P}_{\nu-1} + \lambda_{\nu+1}\mathbf{P}_{\nu+1} + \ldots + \lambda_\rho\mathbf{P}_\rho \tag{11.2}$$

$$\mathbf{S} = \frac{1}{\lambda_1}\mathbf{P}_1 + \ldots + \frac{1}{\lambda_{\nu-1}}\mathbf{P}_{\nu-1} + \frac{1}{\lambda_{\nu+1}}\mathbf{P}_{\nu+1} + \ldots + \frac{1}{\lambda_\rho}\mathbf{P}_\rho \tag{11.3}$$

and they are mutually invertible in the generalized sense /16/.

12. The basic structural formula (4.5) opens up fundamentally new possibilities for classifying elastic solids. Formula (4.5) reduces this problem to that of constructing reasonable classifications of orthogonal decompositions of the space Σ.

13. The proposed approach makes it possible to extend the existing energy condition of elastic Huber-Mises-Hencky behaviour to anisotropic solids. We will introduce the energy of the ν-th elastic eigenstates

$$2E_\nu \equiv \frac{s_\nu^2}{\lambda_\nu} = \frac{\sigma\cdot\mathbf{P}_\nu\cdot\sigma}{\lambda_\nu} \tag{13.1}$$

$$E_1 + \ldots + E_\rho = \Phi \tag{13.2}$$

We will write the energy condition of elastic behaviour in the form

$$F(E_1, \ldots, E_\rho) \leqslant 0 \tag{13.3}$$

We note two special cases. The first case is

$$a_1E_1 + \ldots + a_\rho E_\rho \leqslant k_0^2 \tag{13.4}$$

where $k_0\sqrt{\lambda_\nu/a_\nu}$ can be regarded as the limiting strength for ν-x elastic eigenstates. The second case is

$$E_1 \leqslant t_1, \ldots, E_\rho \leqslant t_\rho \tag{13.5}$$

where t_ν characterizes the limit of elastic behaviour in the subspace Π_ν. Generally speaking, when $E_\nu = t_\nu$, Hooke's Law (7.3) can still hold in the remaining subspaces $\Pi_1, \ldots, \Pi_{\nu-1}, \Pi_{\nu+1}, \ldots, \Pi_\rho$

Note that under the existing generalizations of the Huber-Mises-Hencky condition (see e.g. /19–21/), its quadratic form is retained, but the energy sense is lost. *

*) The energy sense of any quadratic yield condition is explained in a paper published in "Uspekhi Mekhaniki", No.3, 1984.

14. Combining the proposed approach based on the elastic eigenstates with the approach based on symmetry, strengthens both. As usual, we regard as the symmetry group of the tensor $\mathbf{L} \in T$, the subgroup $O(\mathbf{L}) \subset O$, consisting of all orthogonal transformations $\mathbf{X} \to Q*\mathbf{X}$, $\mathbf{Q} \in O$ preserving this tensor

$$O(\mathbf{L}) \equiv \{\mathbf{Q} \in O \mid \mathbf{Q}*\mathbf{L} = \mathbf{L}\} \tag{14.1}$$

Here $\mathbf{Q}$ is an orthogonal, second-rank tensor and the linear operation $\mathbf{Q}*$ is defined on the decomposable tensors by the formula

$$\mathbf{Q}*(\mathbf{a}_1 \otimes \ldots \otimes \mathbf{a}_4) \equiv \mathbf{Q}\mathbf{a}_1 \otimes \ldots \otimes \mathbf{Q}\mathbf{a}_4$$

Using the structural formula (4.5), we obtain the following fundamental theorem on elastic symmetry.

Theorem 3. The symmetry group of an elastic solid $O(\mathbf{C})$ is equal to the intersection of the symmetry groups of its projectors

$$O(\mathbf{C}) = O(\mathbf{P}_1) \cap \ldots \cap O(\mathbf{P}_\rho) \tag{14.2}$$

Proof. If $\mathbf{Q}*\mathbf{P}_\alpha = \mathbf{P}_\alpha$ for all $\alpha = 1, \ldots, \rho$, then we also have

$$\mathbf{Q}*\mathbf{C} = \mathbf{Q}*(\lambda_1\mathbf{F}_1 + \ldots) = \lambda_1\mathbf{Q}*\mathbf{P}_1 + \ldots = \mathbf{C} \tag{14.3}$$

Conversely, let $\mathbf{Q}*\mathbf{C} = \mathbf{C}$. Then

$$\lambda_1\mathbf{Q}*\mathbf{P}_1 + \ldots = \lambda_1\mathbf{P}_1 + \ldots \tag{14.4}$$

Since $\mathbf{P}_1, \ldots, \mathbf{P}_\rho$ is an orthogonal expansion of unity, it follows that so is $\mathbf{Q}*\mathbf{P}_1, \ldots, \mathbf{Q}*\mathbf{P}_\rho$, because $\mathbf{Q}*\mathbf{I} = \mathbf{I}$ for all $\mathbf{Q} \in O$. In accordance with the uniqueness of the structural formula (4.6) we obtain $\mathbf{Q}*\mathbf{P}_\alpha = \mathbf{P}_\alpha$, $\alpha = 1, \ldots, \rho$.

Theorem 3 makes is possible to find all elastic eigenstates of an elastic solid symmetrical with respect to the given subgroup $G \subset O$. Indeed, the process is equivalent to finding all expansions of unity (3.14) in which the terms are symmetrical in G. This in turn implies that we must obtain all decompositions of the space of symmetric second-rank tensors Σ (4.3) in which all terms are invariant with respect to the group G. The solution of this problem leads to a very simple theory of elastic symmetry for solids of any structure.

15. Let us consider two elementary examples.

An isotropic elastic solid. The elastic eigenstates of an isotropic solid have been well known for a long time. They can be represented by ant spherical tensor and any deviator.

The decomposition (4.3) has the following standard form:

$$\Sigma = \Pi \oplus \Delta \tag{15.1}$$

where Π is a 1-dimensional space of spherical tensors and Δ is a 5-dimensional space of deviators. The decomposition (15.1) can be obtained at once from the theorem on symmetry, since Π and Δ are unique subspaces invariant with respect to the whole orthogonal group O. This phrase represents, if you please, the derivation of Hooke's Law for an isotropic solid, of record-breaking brevity (see e.g. /22/). The projectors on Π and Δ are

$$\mathbf{I}_\Pi = {}^1/_3\,\mathbf{1} \otimes \mathbf{1}, \quad \mathbf{I}_\Delta = \mathbf{I} - {}^1/_3\,\mathbf{1} \otimes \mathbf{1}$$

The basic structural formula (4.6) takes the form

$$\mathbf{C} = \lambda_\Pi \mathbf{I}_\Pi + \lambda_\Delta \mathbf{I}_\Delta$$

This is identical with (1.3), and

$$\lambda_\Pi = 3K = 3\lambda + 2\mu, \quad \lambda_\Delta = 2\mu$$

Thus for an isotropic body the simple modulus λ_Π (triple modulus of volume rigidity) and quintuple modulus λ_Δ (doubled shear modulus) represent the true rigidity moduli (9.3). There are no rigidity distributors (9.5) nor aligning angles, since the body does not contain even a single separated fibre. The structural indices of the family of isotropic elastic solids are

$$\langle 1 + 5 \rangle, \quad [2 + 0 + 0]$$

The orthogonal decomposition (7.3) of Hooke's Law takes the form

$$\sigma_\Pi = \lambda_\Pi \varepsilon_\Pi, \quad \sigma_\Delta = \lambda_\Delta \varepsilon_\Delta$$

where

$$\sigma_\Pi = {}^1/_3(\mathrm{tr}\,\sigma)\,\mathbf{1}, \quad \varepsilon_\Pi = {}^1/_3(\mathrm{tr}\,\varepsilon)\,\mathbf{1}$$
$$\sigma_\Delta = \sigma - \sigma_\Pi, \quad \varepsilon_\Delta = \varepsilon - \varepsilon_\Pi$$

represent the sperhical parts and deviators.

Formula (13.2) represents the separation of elastic energy into volume-change energy E_Π and form-change energy E_Δ

$$\Phi = E_{\Pi} + E_{\Delta}$$
$$2E_{\Pi} = \frac{1}{\lambda_{\Pi}} \sigma\cdot \mathbf{I}_{\Pi}\cdot\sigma = \frac{s_{\Pi}^2}{\lambda_{\Pi}} = \frac{1}{6\lambda_{\Pi}} (\mathrm{tr}\,\sigma)^2$$
$$2E_{\Delta} = \frac{1}{\lambda_{\Delta}} \sigma\cdot \mathbf{I}_{\Delta}\cdot\sigma = \frac{s_{\Delta}^2}{\lambda_{\Delta}} = \frac{1}{\lambda_{\Delta}} \left[\sigma\cdot\sigma - \frac{1}{3} (\mathrm{tr}\,\sigma)^2\right]$$

Putting in (13.5) $t_{\Pi} = \infty$, we obtain the Huber-Mises-Hencky condition $E_{\Delta} \leqslant t_{\Delta}$ of elastic behaviour.

It is interesting to note that a definition of an isotropic solid itself can be given without mentioning its symmetry. In fact, the following theorem holds: an elastic solid **C** is isotropic if and only if any simple shear

$$\omega = \tau(\mathbf{m}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{m}),\ \mathbf{mn} = 0 \tag{15.2}$$

represents its elastic eigenstate. The non-triviality of this proposition lies in the fact that the condition of equality of the rigidity moduli is not previously assumed for the shears (15.2). The proof is given in /23/.

Isotropic elastic solids with zero Poisson's ratio

$$\nu \equiv \frac{\lambda_{\Pi} - \lambda_{\Delta}}{2\lambda_{\Pi} + \lambda_{\Delta}} = 0$$

form a special family. We shall call them perfect elastic solids. We have for them $\lambda_{\Pi} = \lambda_{\Delta} = \lambda$, and Hooke's Law has the form

$$\mathbf{C} = \lambda \mathbf{I}, \text{ i.e. } \quad \sigma = \lambda\varepsilon$$

For an ideal material any state is the elastic eigenstate. The structural indices are limiting

$$\langle 6\rangle,\ [1+0+0] \tag{15.3}$$

Let us now consider a more interesting case.

A transversally isotropic elastic solid. Suppose an elastic solid is symmetric with respect to the group of rotations $R_{\mathbf{k}}$ about the unit vector $\mathbf{k}$, i.e. $O(\mathbf{C}) \supset R_{\mathbf{k}}$. According to (14.2) we must seek all decompositions (4.3) in which the terms will be stable with respect to the group $R_{\mathbf{k}}$. Let us choose the Cartesian basis in such a manner, that the third axis is directed along $\mathbf{k}$. The solution is

$$\Sigma = \Pi_1 \oplus \Pi_2 \oplus \Pi_3 \oplus \Pi_4 \tag{15.4}$$

Here Π_1 is a 1-dimensional space of axisymmetric tensors of the form

$$\alpha_1 \sim p \begin{Vmatrix} \cos\varkappa & 0 & 0 \\ 0 & \cos\varkappa & 0 \\ 0 & 0 & \sqrt{2}\sin\varkappa \end{Vmatrix} \tag{15.5}$$

Π_2 is a 1-dimensional space of axisymmetric tensors of the form

$$\alpha_2 \sim q \begin{Vmatrix} \sin\varkappa & 0 & 0 \\ 0 & \sin\varkappa & 0 \\ 0 & 0 & -\sqrt{2}\cos\varkappa \end{Vmatrix} \tag{15.6}$$

Π_3 is a 2-dimensional space of simple shears of the form

$$\alpha_3 \sim \begin{Vmatrix} 0 & 0 & u \\ 0 & 0 & v \\ u & v & 0 \end{Vmatrix} \tag{15.7}$$

and Π_4 is a 2-dimensional space of simple shears of the form

$$\alpha_4 \sim \begin{Vmatrix} x & y & 0 \\ y & -x & 0 \\ 0 & 0 & 0 \end{Vmatrix} \tag{15.8}$$

The formula

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

expresses an arbitrary tensor α is terms of six parameters p, q, u, v, x, y.

Fig.2 shows the elastic eigenstates of a transversally isotropic solid $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. The reader is invited to confirm that $\alpha_{\nu}\cdot\alpha_{\mu} = 0$ when $\nu \neq \mu$ and, after arbitrary rotation about the unit vector k, any tensor $\alpha_{\nu} \in \Pi_{\nu}$ remains in Π_{ν}.

The decomposition (15.4) is intuitively clear. Let us imagine a transversally isotropic solid in the form of an isotropic matrix reinforced with fibres in the $\mathbf{k}$ direction. First acquaintance with the resistance of materials to small deformations is sufficient to enable us to assert that simple shears (15.7) and (15.8) will represent the elastic eigenstates of the solid in question. Indeed, under the shears (15.7), (15.8) the work performed by the isotropic matrix is the only work done. This explains the presence of two-dimensional terms Π_3, Π_4 in (15.4). As regards the pair Π_1, Π_2 with variable $\varkappa$, it represents an arbitrary orthogonal decomposition of the two-dimensional space $(\Pi_3 \oplus \Pi_4)^{\perp}$.

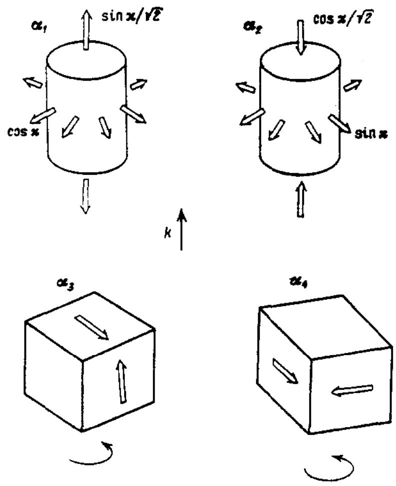

Fig.2

The projectors $P_1(\varkappa, k)$, $P_2(\varkappa, k)$, $P_3(k)$, $P_4(k)$ can be written down using (3.13).

Hooke's Law has the form

$$\sigma_1 = \lambda_1 \varepsilon_1, \ldots, \sigma_4 = \lambda_4 \varepsilon_4$$

The system of elastic parameters defining a concrete, transversally elastic body, consists of

1) 4 true rigidity moduli $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ where λ_3, λ_4 can be regarded as shear moduli;

2) rigidity distributor $\varkappa$, $0 < \varkappa \leqslant \pi/2$;

3) two angles θ, φ defining the position of the axis of symmetry relative to the laboratory coordinate system.

The structural indices are

$$\langle 1+1+2+2 \rangle, \ [4+1+2] \tag{15.9}$$

The elastic constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \varkappa$ can be expressed in terms of five non-zero components C_{ijkl} different from each other, on the Cartesian basis used.

The following theorem holds: an elastic solid is transversally isotropic if and only if there is a unit vector $\mathbf{k}$ such that the simple shears

$$\mathbf{a} \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{a}, \quad \mathbf{ak} = 0$$
$$\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a}, \quad \mathbf{ab} = \mathbf{ak} = \mathbf{bk} = 0$$

are its elastic eigenstates. The last two theorems indicate the remarkable role played by simple shears in problems of elastic behaviour for small deformations.

Let us consider a completely different case.

A completely asymmetric elastic solid. Let us take an orthonormalized triad of axisymmetric deviators

$$\boldsymbol{\tau}_1 = \frac{\sqrt{6}}{6}(\mathbf{1} - 3\mathbf{t}_1 \otimes \mathbf{t}_1)$$
$$\boldsymbol{\tau}_2 = \frac{\sqrt{6}}{6}(\mathbf{1} - 3\mathbf{t}_2 \otimes \mathbf{t}_2)$$
$$\boldsymbol{\tau}_3 = \frac{\sqrt{6}}{6}(\mathbf{1} - 3\mathbf{t}_3 \otimes \mathbf{t}_3)$$

where $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ is a triad of equally inclined unit vectors

$$\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_3 = \mathbf{t}_3\mathbf{t}_1 = \frac{\sqrt{3}}{3}$$

It can be confirmed that indeed $\boldsymbol{\tau}_\alpha \cdot \boldsymbol{\tau}_\beta = \delta_{\alpha\beta}$. Let us introduce a two-dimensional orthogonal projector

$$\mathbf{P} = \mathbf{1} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1} - (\boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 \otimes \boldsymbol{\tau}_2 + \boldsymbol{\tau}_3 \otimes \boldsymbol{\tau}_3)$$

and consider the family of elastic solids

$$\mathbf{C} = \frac{\lambda}{3}\mathbf{1} \otimes \mathbf{1} + \mu_1 \boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_1 + \mu_2 \boldsymbol{\tau}_2 \otimes \boldsymbol{\tau}_2 + \mu_3 \boldsymbol{\tau}_3 \otimes \boldsymbol{\tau}_3 + \mu_4 \mathbf{P} \tag{15.10}$$

whose structural indices are

$$\langle 1+1+1+1+2 \rangle, \ [5+0+3]$$

If the true rigidity moduli $\lambda, \mu_1, \mu_2, \mu_3, \mu_4$ are pairwise different, then from Theorem (14.2) we find that

$$O(\mathbf{C}) = O(\boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_1) \cap O(\boldsymbol{\tau}_2 \otimes \boldsymbol{\tau}_2) \cap O(\boldsymbol{\tau}_3 \otimes \boldsymbol{\tau}_3) = \{\mathbf{1}, -\mathbf{1}\}$$

i.e. the solid is asymmetric in the limit (a triclinic system). At the same time the solids (15.10) possess a well-defined mathematical structure. Firstly, they are spatially isotropic, i.e. for a spherical deformation $\boldsymbol{\varepsilon} = \varepsilon\mathbf{1}$ a corresponding hydrostatic stress state $\boldsymbol{\sigma} = \mathbf{C} \cdot \boldsymbol{\varepsilon} = (\lambda\varepsilon)\mathbf{1}$ exists. Secondly, the material tensor reference frame contains here a unique configuration of three axisymmetric deviators.

This somewhat artificial example was given to illustrate the wealth of possibilities of elastic behaviour contained within the structural formulas obtained.

16. Everything discussed here concerns the behaviour of linearly elastic solids. A simple, though very wide and apparently important practical class of non-linearly elastic solids will be described by defining equations of the type

$$\boldsymbol{\sigma} = \mathbf{C}(\boldsymbol{\varepsilon}) \cdot \boldsymbol{\varepsilon}$$

where the "rigidity tensor" $\mathbf{C}(\boldsymbol{\varepsilon})$ is a function of deformations of the form

$$\mathbf{C}(\boldsymbol{\varepsilon}) = \lambda_1(\boldsymbol{\varepsilon})\mathbf{P}_1(\boldsymbol{\varepsilon}) + \ldots + \lambda_\rho(\boldsymbol{\varepsilon})\mathbf{P}_\rho(\boldsymbol{\varepsilon})$$

the system $\mathbf{P}_1(\boldsymbol{\varepsilon}), \ldots, \mathbf{P}_\rho(\boldsymbol{\varepsilon})$ represents, for any $\boldsymbol{\varepsilon}$, an orthogonal decomposition of unity

$$\mathbf{P}_1(\boldsymbol{\varepsilon}) + \ldots + \mathbf{P}_\rho(\boldsymbol{\varepsilon}) = \mathbf{I} \qquad (16.1)$$
$$\mathbf{P}_\alpha(\boldsymbol{\varepsilon})\circ\mathbf{P}_\alpha(\boldsymbol{\varepsilon}) = \mathbf{P}_\alpha(\boldsymbol{\varepsilon}), \quad \mathbf{P}_\beta(\boldsymbol{\varepsilon})\circ\mathbf{P}_\alpha(\boldsymbol{\varepsilon}) = 0, \quad \alpha \neq \beta$$

and $\lambda_\alpha(\boldsymbol{\varepsilon})$ are invariants of some subgroup $G \subset O$.

In particular, we may consider a situation in which the decomposition (16.1) is constant and the "rigidity moduli" are the only parameters depending on the deformations

$$\mathbf{C}(\boldsymbol{\varepsilon}) = \lambda_1(\boldsymbol{\varepsilon})\mathbf{P}_1 + \ldots + \lambda_\rho(\boldsymbol{\varepsilon})\mathbf{P}_\rho$$

An example of such a dependence is

$$\mathbf{C}(\boldsymbol{\varepsilon}) = \lambda_\Pi \mathbf{I}_\Pi + \lambda_\Delta(\boldsymbol{\varepsilon})\mathbf{I}_\Delta$$

If $\lambda_\Delta(\boldsymbol{\varepsilon})$ is invariant, then the body is isotropic. This case serves as the basis for the theory of small elastoplastic deformations /24/.

17. We have described how an elastic solid is constructed mathematically. The results obtained do not "alter" the usual form (1.1) of the law of elasticity; on the contrary, in a number of situations it is unsurpassed in its simplicity.

The development of the approaches connecting an open mathematical structure of any linearly elastic solid with the physical structure of a specific solid (crystal, polycrystal, a composite, alloy, plastic, glass, etc.) as well as the development of experimental procedures aimed at determining the true rigidity moduli and the elastic eigenstates, all represent intersecting and complex problems requiring a separate investigation.

A preliminary account of the results obtained was given in /25/. In /26/ the approach was applied to thermoelasticity, and in a paper referred to there, to the theory in the strength of materials.

In conclusion we stress, that the results given here are as accurate, and, within that accuracy, as general, as law (1.2) itself. We have merely polished a few new faces of a crystal discovered by Robert Hooke,

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Appendix. Below we give a list of relations enabling any formula of this paper to be rapidly transformed into Cartesian index language

$$\boldsymbol{\alpha}, \mathbf{C}, \mathbf{1}, \mathbf{n} \leftrightarrow \alpha_{ij}, C_{ijkl}, \delta_{ij}, n_i$$
$$\mathbf{nm}, \mathbf{n}\otimes\mathbf{m} \leftrightarrow n_i m_i, n_i m_j$$
$$\boldsymbol{\alpha\beta}, \boldsymbol{\alpha}\otimes\boldsymbol{\beta} \leftrightarrow \alpha_{ik}\beta_{kj}, \alpha_{ij}\beta_{kl}$$
$$\boldsymbol{\alpha}^2, \boldsymbol{\alpha}^3 \leftrightarrow \alpha_{ik}\alpha_{kj}, \alpha_{ik}\alpha_{kl}\alpha_{lj}$$
$$\operatorname{tr}\boldsymbol{\alpha}, |\boldsymbol{\alpha}| \leftrightarrow \alpha_{ii}, (\alpha_{ij}\alpha_{ij})^{1/2}$$
$$\boldsymbol{\alpha}\cdot\boldsymbol{\beta}, \mathbf{n}\boldsymbol{\alpha}\mathbf{m} \leftrightarrow \alpha_{ij}\beta_{ij}, \alpha_{ij}n_i m_j$$
$$\mathbf{C}\cdot\boldsymbol{\alpha}, \boldsymbol{\alpha}\cdot\mathbf{C}\cdot\boldsymbol{\beta} \leftrightarrow C_{ijkl}\alpha_{kl}, C_{ijkl}\alpha_{ij}\beta_{kl}$$
$$\mathbf{C}\circ\mathbf{S}, \mathbf{C}\cdot\mathbf{L} \leftrightarrow C_{ijpq}S_{pqkl}, C_{ijkl}L_{ijkl}$$
$$\mathbf{Q} * \mathbf{C} \leftrightarrow Q_{ip}Q_{jq}Q_{kr}Q_{ls}C_{pqrs}$$
$$\mathbf{I}_{ijkl} = {}^1/_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$$

We rewrite, as examples, three principal formulas, namely

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad (1.1)$$
$$C_{ijkl} = \lambda_1 P_{(1)\,ijkl} + \ldots + \lambda_\rho P_{(\rho)\,ijkl} \qquad (4.6)$$
$$C_{ijkl} = \lambda_I \omega_{(I)\,ij}\,\omega_{(I)\,kl} + \ldots + \lambda_6 \omega_{(6)\,ij}\omega_{(6)\,kl} \qquad (4.9)$$

Note added in proof. The author discovered in the library a copy of the Philosophical Transactions of the Royal Society of London, of 1856, containing a paper by the future Lord Kelvin: Thomson W. Elements of a Mathematical Theory of Elasticity, p.481-498. Kelvin introduces in this paper the elastic eigenvalues, calling them the Six Principal Strain-Types of the body. He could not possibly obtain the structural formulas, since the mathematical techniques necessary did not exist at the time. Moreover, the author discovered that the paper was reviewed with scepticism in /2/ at the end of last century, and then promptly forgotten. It therefore seems appropriate to call the true rigidity moduli the Kelvin moduli.

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